Some aspects of colour superconductivity in finite systems

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Outline

- § Introduction
- § The model
- § Finite size effects / color and baryon number projection
- § Results
- § Conclusions

An example

$$\hat{H} = \sum_i \epsilon_i \; \hat{a}_i^\dagger \; \hat{a}_i - \; g \; \sum_i \hat{a}_i^\dagger \; \hat{a}_{\bar{i}}^\dagger \; \hat{a}_{\bar{j}} \; \hat{a}_j \quad ,$$

and $\hat{a}$ are fermionic operators.

In the absence of interaction ($g \to 0$) the ground state is the **Fermi sea**:

$$|k_F\rangle = \hat{a}_1^\dagger \; \hat{a}_2^\dagger \; \ldots \; \hat{a}_N^\dagger \; |0\rangle$$

$$n_i \equiv \langle k_F | \hat{a}_i^\dagger \; \hat{a}_i | k_F \rangle = \theta(N - i) \quad ; \quad \rho_{ij} \equiv \langle k_F | \hat{a}_i \; \hat{a}_j | k_F \rangle = 0$$

No perturbative calculation can give $\rho_{ij} \neq 0$ starting from $|k_F\rangle$!

However:

the Fermi surface is unstable with respect to an attractive interaction among the fermions

we can approximate the ground state with the **BCS** state:

$$|BCS\rangle = \prod_i \left[u_i + v_i \ \hat{a}_i^\dagger \ \hat{a}_{\bar{i}}^\dagger \right] \ |k_F\rangle$$

where

$$|u_i|^2 + |v_i|^2 = 1$$

A non-vanishing **order parameter** caractherizes this state:

$$\rho_{\bar{i}i} = u_i \ v_i \neq 0$$

QCD at large density

- because of asymptotic freedom interactions among quarks are weak at large densities
- the quarks would fill all the states up to the Fermi level
- Is there an attractive interaction among quarks?
  - one-gluon exchange
  - instanton mediated interaction
- Cooper pairs of quarks form and the corresponding order parameter is non-vanishing: $\langle q\ q \rangle \neq 0$
- there are many degrees of freedom: many possible states can form

Two flavours ($N_f = 2$) and three colours ($N_c = 3$)

- Order parameter

$$\langle \psi_{ri\alpha} \, \psi_{sj\beta} \rangle \propto \delta_{rs} \, \epsilon_{ij} \, \epsilon_{\alpha\beta 3}$$

- Symmetry breaking pattern:

$$SU(3)_C \to SU(2)_C$$

- 5 gluons acquire mass, 3 gluons remain massless and one of them mixes with the photon.

The model[a]

- We use a hamiltonian

$$\mathbf{H} = \mathbf{H_0} + \mathbf{H_{int}}$$

where $\mathbf{H_0}$ is the Dirac hamiltonian

$$\mathbf{H_0} = -\int \mathbf{d^3x}\, \overline{\psi}_{\mathbf{ri}\alpha}(\mathbf{x})\ \mathbf{i}\vec{\alpha}\cdot\vec{\nabla}\ \psi_{\mathbf{ri}\alpha}(\mathbf{x})$$

and $\mathbf{H_{int}}$ is the interaction hamiltonian

$$\mathbf{H_{int}} = -\mathcal{K}\ \mathbf{\Xi}_{\mathbf{kl};\alpha\beta\gamma\delta} \times$$

$$\int d^3x\ \overline{\psi}_{R1\alpha}(x)\ \psi_{Lk\gamma}(x)\overline{\psi}_{R2\beta}(x)\ \psi_{Lk\delta}(x) + h.c.$$

[a] M. Alford, K. Rajagopal and F. Wilczek, Phys. Lett. B422,247 (1998)

- ψ = quark field

- $(i, j, k, l, \ldots)$ = flavour
- colour $(\alpha, \beta, \gamma, \delta, \ldots)$ = colour
- (R, L) = helicity

- $\Xi_{kl;\alpha\beta\gamma\delta} \equiv \epsilon_{kl} \ (3\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma})$

At **low** density:

$$|\Psi\rangle = |k_F\rangle = \prod_{\alpha i \vec{k}} \hat{a}^\dagger_{Li\alpha}(\vec{k})\ \hat{a}^\dagger_{Ri\alpha}(\vec{k})|0\rangle$$

At **high** density:

$$|\Psi\rangle = \hat{G}_L\ \hat{G}_R\ |k_F\rangle$$

$$\hat{G}_L \equiv \prod_{\alpha\beta\vec{k}} \left[\mathbf{u_{AL}(k)} + \epsilon_{\alpha\beta 3}\ \mathbf{v_{AL}(k)}\ \hat{a}^\dagger_{L1\alpha}(\vec{k})\ \hat{a}^\dagger_{L2\beta}(-\vec{k})\right]$$

$$\cdot \prod_{\alpha\beta\vec{k}} \left[\mathbf{u_{BR}(k)} + \epsilon_{\alpha\beta 3}\ \mathbf{v_{BR}(k)}\ \hat{b}^\dagger_{R1\alpha}(\vec{k})\ \hat{b}^\dagger_{R2\beta}(-\vec{k})\right]$$

$$\cdot \prod_{\alpha\beta\vec{k}} \left[\mathbf{u_{CR}(k)} + \epsilon_{\alpha\beta 3}\ \mathbf{v_{CR}(k)}\ \hat{c}^\dagger_{R1\alpha}(\vec{k})\ \hat{c}^\dagger_{R2\beta}(-\vec{k})\right]$$

where $u_{AL}(k) \equiv \cos\theta_{AL}(k) \quad , \quad v_{AL}(k) \equiv \sin\theta_{AL}(k)\ e^{i\xi_{AL}(k)}$

We minimize the thermodynamic potential

$$F\,(\theta,\xi) = \langle\Psi|\hat{H} - \mu\hat{N}|\Psi\rangle$$

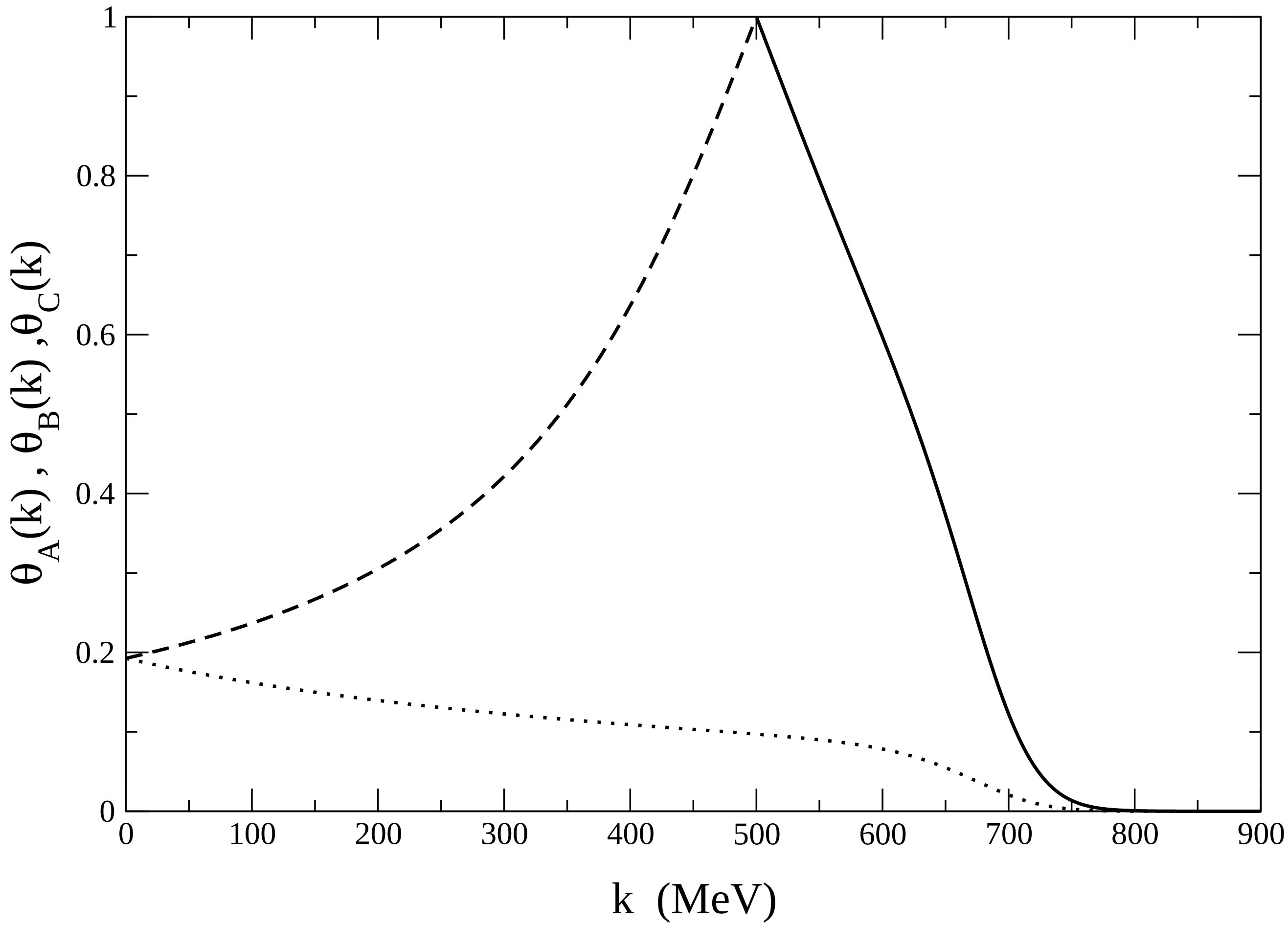
and we obtain the gap equation

$$\begin{aligned}\Delta &= \frac{\mathcal{K}}{\pi^2}\int_0^\infty dk\; k^2\;\mathcal{F}(k)^2\;\left[\theta(k-k_F)\;\sin 2\theta_A(k)\right.\\ &+ \left.\sin 2\theta_B(k) + \theta(k_F-k)\;\sin 2\theta_C(k)\right]\end{aligned}$$

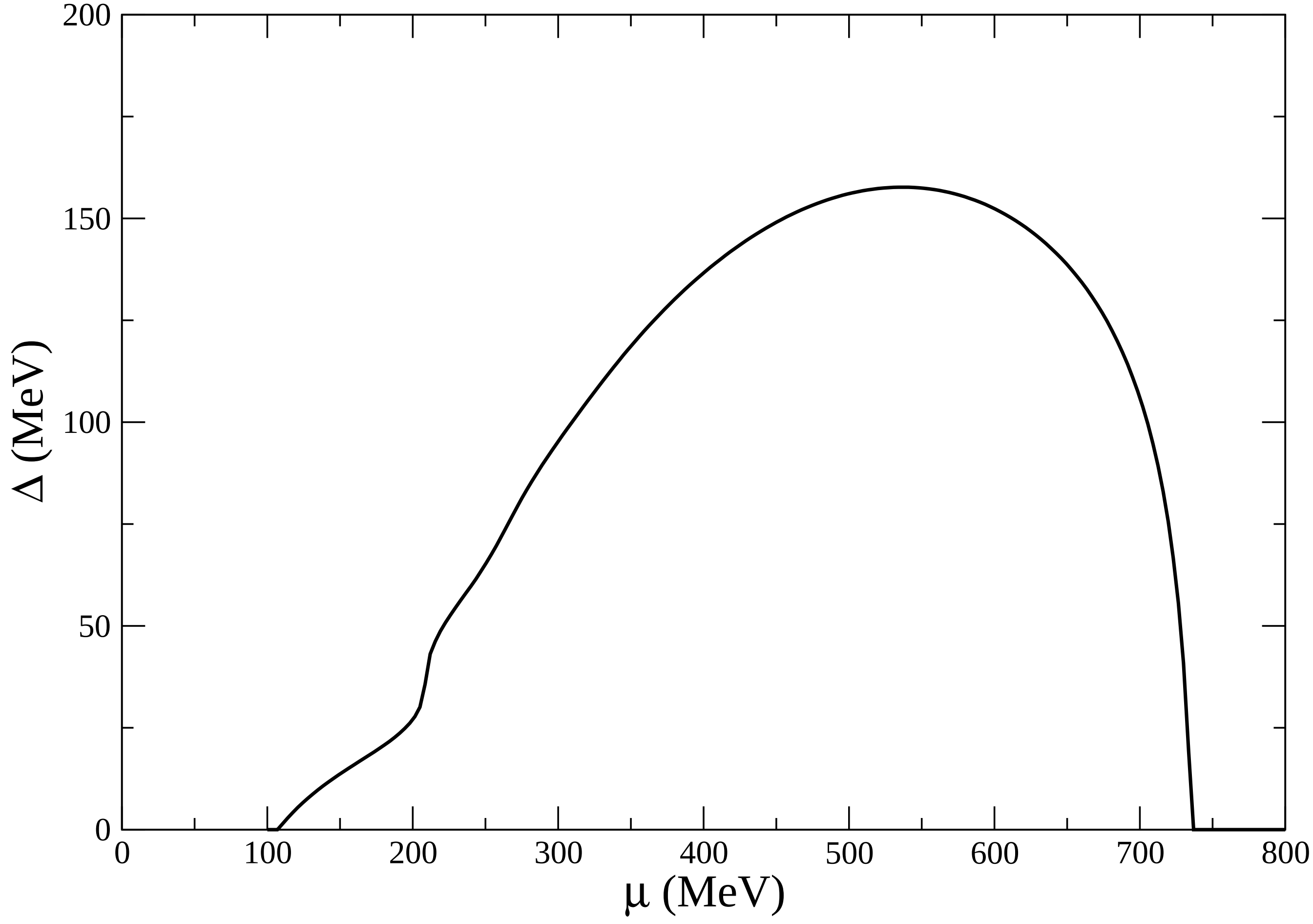
where

$$\begin{aligned}\tan 2\;\theta_{A,B,C}(k) &= \mathcal{F}(k)^2\;\Delta\\ &\times \left\{\frac{\theta(k-k_F)}{k-\mu},\frac{1}{k+\mu},\frac{\theta(k_F-k)}{\mu-k}\right\}\end{aligned}$$

$\mathcal{F}(k)\;$ = form factor



Variational solutions for the **2SC**



Gap for the **2SC**

Finite systems[a]

What happens to the Color Superconductor in a finite system?

Need to address three problems:

- we are considering "chunks" of quark matter at large density (boundary conditions for the quarks)
- the BCS state does not have definite baryon number
  baryon number is fixed only on average
- the BCS state is not a color singlet
  there is a preferred color direction

[a] P. Amore, M. Birse, J. McGovern and N. Walet, Phys. Rev. D65:074005,2002

Projection over baryon number

The **BCS** state does not have a fixed baryon number: N_B is fixed only *on average* through the chemical potential.

We introduce a **"rotated" BCS state** ($\zeta = e^{i\phi}$):

$$|\Psi(\zeta)\rangle \equiv \mathbf{e}^{\frac{\mathbf{i}}{\mathbf{2}}\left(\hat{\mathbf{N}}-\mathbf{N}\right)\,\phi}|\Psi\rangle = \zeta^{\left(\hat{\mathbf{N}}-\mathbf{N}\right)/\mathbf{2}}\,|\Psi\rangle$$

with the rotated BCS operator

$$\hat{G}_L(\zeta) \equiv \zeta^{\hat{\mathbf{N}}/\mathbf{2}}\,\hat{G}_L\,\zeta^{-\hat{\mathbf{N}}/\mathbf{2}}$$

$$= \prod_{\alpha\beta\vec{k}}\left[u_{AL}(k) + \zeta\,\epsilon_{\alpha\beta 3}\,v_{AL}(k)\,\hat{a}^\dagger_{L1\alpha}(\vec{k})\,\hat{a}^\dagger_{L2\beta}(-\vec{k})\right]$$

$$\cdot \prod_{\alpha\beta\vec{k}}\left[u_{BR}(k) + \zeta^*\,\epsilon_{\alpha\beta 3}\,v_{BR}(k)\,\hat{b}^\dagger_{R1\alpha}(\vec{k})\,\hat{b}^\dagger_{R2\beta}(-\vec{k})\right]$$

$$\cdot \prod_{\alpha\beta\vec{k}}\left[u_{CR}(k) + \zeta^*\,\epsilon_{\alpha\beta 3}\,v_{CR}(k)\,\hat{c}^\dagger_{R1\alpha}(\vec{k})\,\hat{c}^\dagger_{R2\beta}(-\vec{k})\right]$$

Expansion

$$|\Psi\rangle = \sum_{k=0}^{\infty} c_k \, \zeta^k \times \text{operators which have a baryon number k}$$

Notice that

- c_k depend upon the variational parameters

- different combinations of operators lead to the very same ζ dependence (for example, $\hat{a}^\dagger \hat{a}^\dagger$, $\hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{b}^\dagger \hat{b}^\dagger$, $\ldots$)

Projected BCS state:

$$|\Psi_n\rangle = \mathcal{C}_n \oint \frac{d\zeta}{\zeta^{n+1}} \, |\Psi(\zeta)\rangle$$

n is the net number of pairs in the condensate (number of quark pairs minus number of antiquark and hole pairs)

Example: the energy

$$\langle \Psi_n | \hat{H} | \Psi_n \rangle \;=\; -2\,\pi\,i\,|C_n|^2 \oint \frac{d\zeta}{\zeta^{n+1}} \langle \Psi(0) | \hat{H} | \Psi(\zeta) \rangle$$

Remark : we cannot apply Wick's theorem directly to calculate the expectation values of operators in the state $|\Psi_n\rangle$.

Solution: Thouless theorem

$$|\Psi(\zeta)\rangle = \hat{S}(\zeta)\;|\Psi(0)\rangle$$

$$\langle \Psi(0) | \hat{O}\;|\Psi(\zeta)\rangle = \langle \Psi(0) | \hat{O}\;\hat{S}(\zeta) | \Psi \rangle$$

Projection over color

- Color is global in this model
- The **2SC** picks a preferred direction in the color space

Solution:

$$|\tilde{\Psi}_n\rangle = \int d\Omega_g \, \hat{\mathcal{U}}_g \, |\Psi_n\rangle$$

- $\hat{\mathcal{U}}_g \in SU(3)_c$ and is parametrized by 8 angles
- $d\Omega_g$ is the volume element of the $SU(3)_c$ manifold

$$d\Omega_g = \frac{1}{2^8} \sin(\phi_b) \sin(\phi_b') \sin(\phi_d) \sin^2\left(\frac{\phi_d}{2}\right) \times d\phi_a \, d\phi_b \, d\phi_c \, d\phi_d \, d\phi_a' \, d\phi_b' \, d\phi_c' \, d\phi_e$$

- $g \in SU(3)$

$$\mathbf{g} = e^{i\phi_a/2\lambda_3}\ e^{i\phi_b/2\lambda_2}\ e^{i\phi_c/2\lambda_3}\ e^{i\phi/2\lambda_5}\ e^{i\phi'_a/2\lambda_3} \times e^{i\phi'_b/2\lambda_2}\ e^{i\phi'_c/2\lambda_3}\ e^{i\phi_e/2\lambda_8}$$

- The transition operators which act on the $2SC$ are:

$$\hat{Q}_a = \int d^3x\ \hat{\psi}^\dagger(x)\ \frac{\lambda_a}{2}\ \hat{\psi}(x)$$

- In principle we need to perform a **8 dimensional integral!**

- The transition operators which act on the $2SC$ are:

$$\hat{\mathbf{Q}}_{\mathbf{1,2,3}}|\Psi_n\rangle \quad = \quad 0$$

- Operator $\hat{\mathbf{Q}}_{\mathbf{8}}$

$$\hat{\mathbf{Q}}_{\mathbf{8}} \quad = \quad \int d^3x \; \hat{\psi}^\dagger(x) \; \frac{\lambda_8}{2} \; \hat{\psi}(x) = \frac{1}{2}\left[\hat{N}_{11} + \hat{N}_{22} - 2\;\hat{N}_{33}\right]$$

- $\hat{\mathbf{Q}}_{\mathbf{5}}$ is the only operator which acts non-trivially (connects one of the paired colors to the unpaired one)

$$d\Omega_5 = \sin^3\frac{\phi}{2}\; d\sin\frac{\phi}{2}$$

- $\langle\tilde{\Psi}_0|\hat{O}|\tilde{\Psi}_0\rangle = \frac{\int d\Omega_5 \ \langle\Psi_0|\hat{O} \ e^{i \ \frac{\phi}{2} \ \hat{\mathbf{Q}}_\mathbf{5}}|\Psi_0\rangle}{\int d\Omega_5 \ \langle\Psi_0|e^{i \ \frac{\phi}{2} \ \hat{\mathbf{Q}}_\mathbf{5}}|\Psi_0\rangle}$
- color singlet state

$$|\tilde{\Psi}_0\rangle = \tilde{C}_0 \ \oint \frac{d\zeta}{\zeta} \ \int d\Omega_5 \ e^{i\phi/2 \ \hat{Q}_5}|\Psi(\zeta)\rangle$$

$$= \tilde{C}_0 \ \oint \frac{d\zeta}{\zeta} \ \int d\Omega_5 \ \tilde{G}_L(\zeta,\phi) \ \tilde{G}_R(\zeta,\phi)|k_F\rangle$$

- color/number "rotated" operators

$$\tilde{G}_R(\zeta,\phi) = \hat{U}_g \ G_R(\zeta) \ \hat{U}_g^\dagger$$

- Thouless theorem

$$e^{i\frac{\phi}{2} \ \hat{\mathbf{Q}}_\mathbf{5}} \ |\Psi(\zeta)\rangle = \hat{W}(\zeta,\phi) \ |\Psi\rangle$$

$$\langle\hat{O}\rangle = -2\ \pi\ i\ |\tilde{C}_0|^2 \int d\Omega_5 \oint \frac{d\zeta}{\zeta}\ \langle\Psi|\hat{O}\ W(\zeta,\phi)|\Psi\rangle$$

For example...

$$\langle\Psi_n|\hat{H}_{int}|\Psi_n\rangle = -\frac{8K}{\Omega d_n}\sum_{\vec{k}}\sum_{\vec{k}'} F^2(k)F^2(k')$$

$$\times\Big\{\theta(k_F-k)\theta(k_F-k')\sin^2\frac{\theta_{kk'}}{2}\ \sin 2\theta_C(k)\sin 2\theta_C(k')\ \mathcal{J}_{b,n}(\theta_C(k),\theta_C(k'))$$

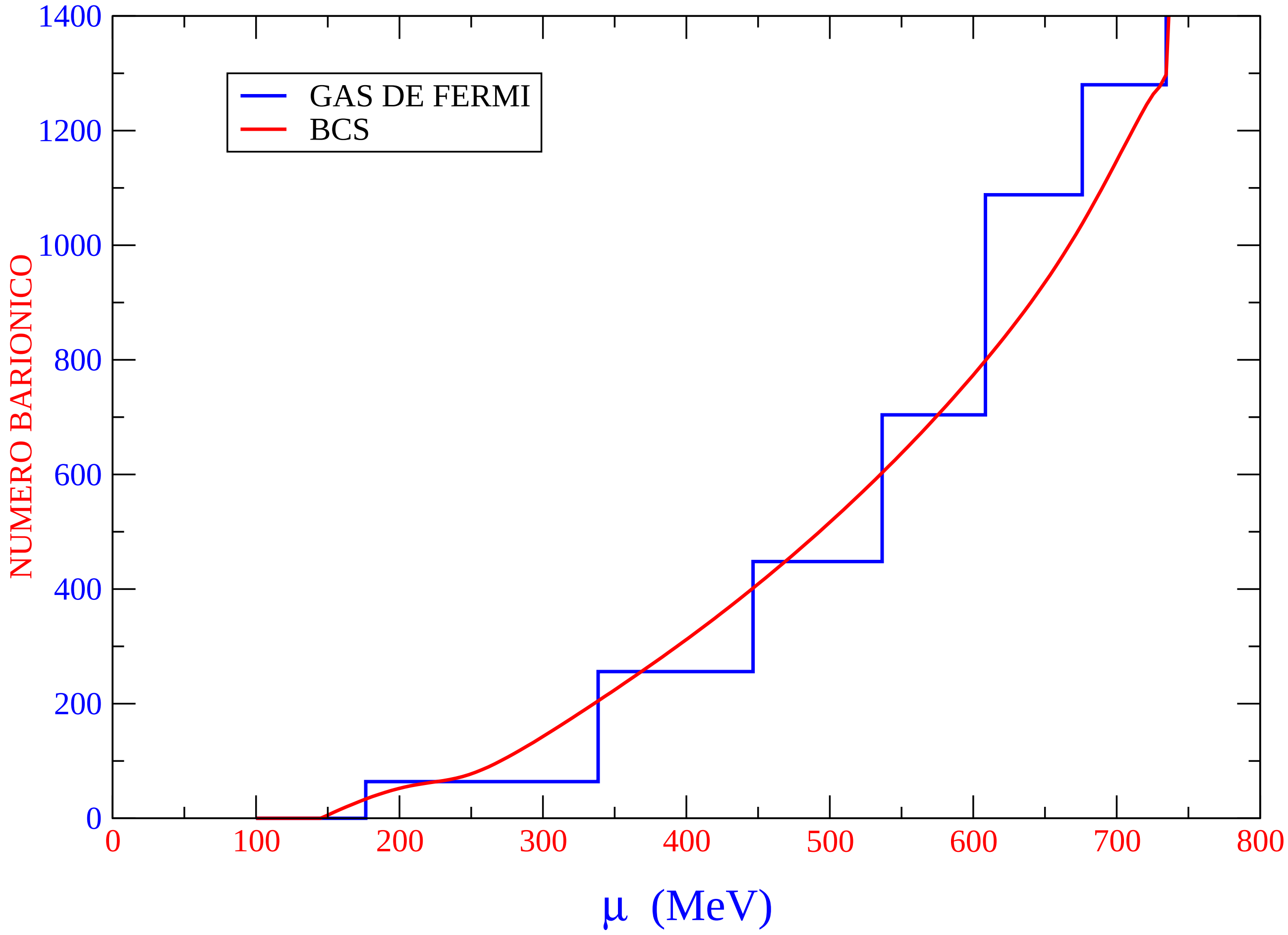
$$+\quad 2\theta(k_F-k)\cos^2\frac{\theta_{kk'}}{2}\sin 2\theta_C(k)\ \sin 2\theta_B(k')\ \mathcal{J}_{b,n}(\theta_C(k),\theta_B(k'))$$

$$+\ 2\theta(k_F-k)\theta(k'-k_F)\cos^2\frac{\theta_{kk'}}{2}\ \sin 2\theta_C(k)\sin 2\theta_A(k')\ \mathcal{J}_{c,n}(\theta_C(k),\theta_A(k'))$$

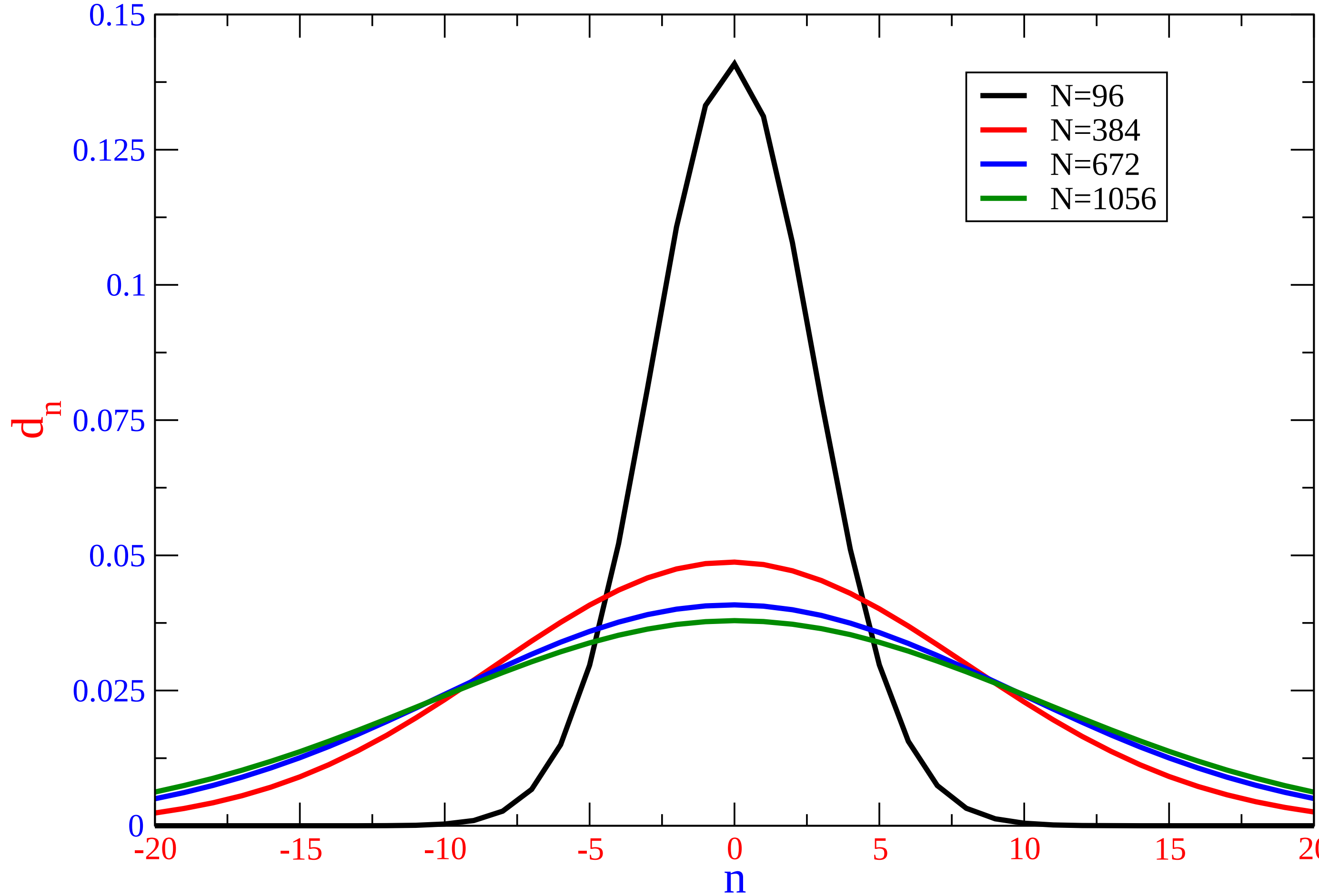
$$+\quad \sin^2\frac{\theta_{kk'}}{2}\sin 2\theta_B(k)\sin 2\theta_B(k')\ \mathcal{J}_{b,n}(\theta_B(k),\theta_B(k'))$$

$$+\ 2\theta(k'-k_F)\sin^2\frac{\theta_{kk'}}{2}\sin 2\theta_B(k)\sin 2\theta_A(k')\ \mathcal{J}_{c,n}(\theta_B(k),\theta_A(k'))$$

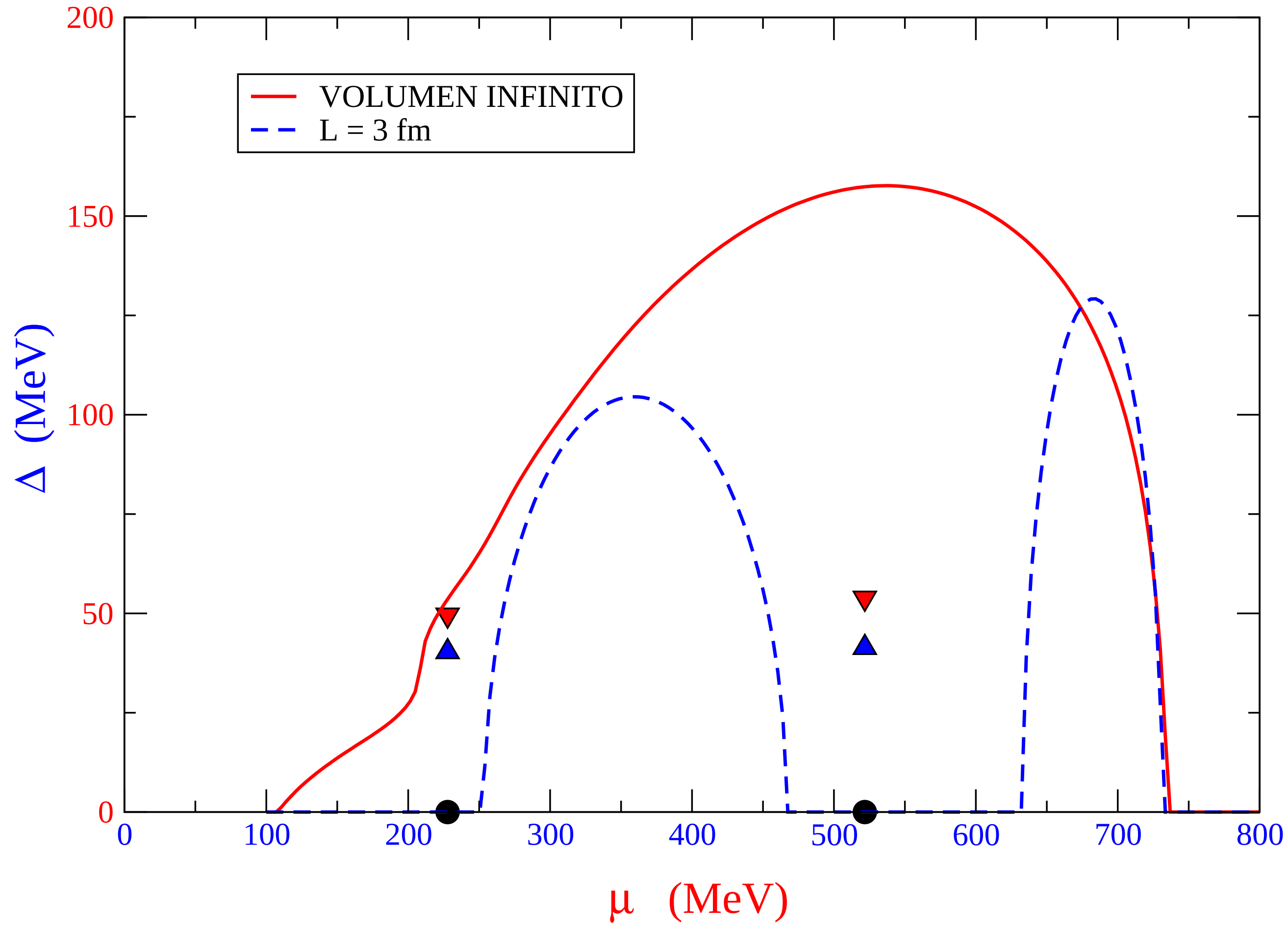
$$+\ \theta(k-k_F)\theta(k'-k_F)\sin^2\frac{\theta_{kk'}}{2}\sin 2\theta_A(k)\sin 2\theta_A(k')\ \mathcal{J}_{a,n}(\theta_A(k),\theta_A(k'))\Big\}$$



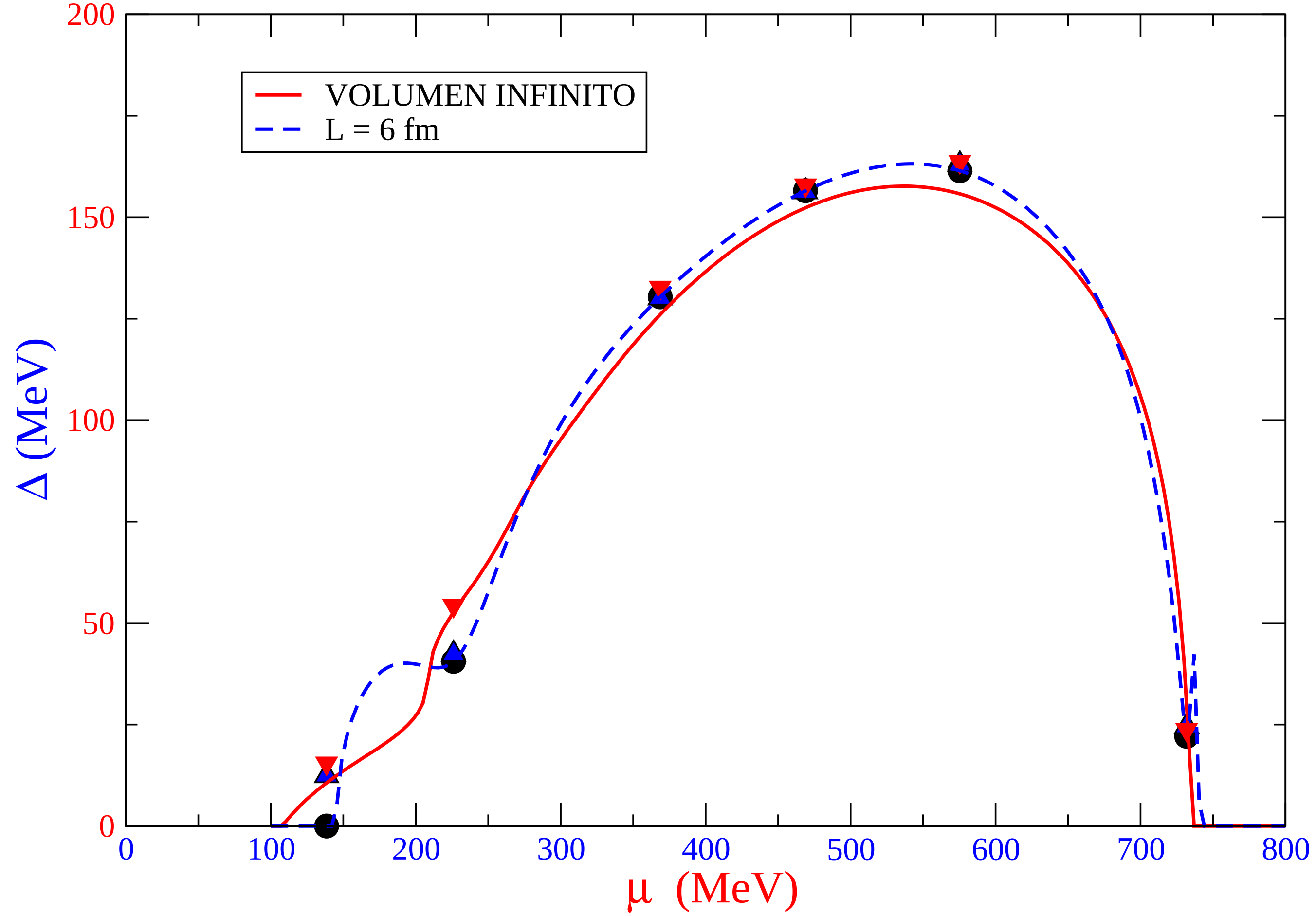
Baryon number of the Fermi sea (black) and of the 2SC (red). $L = 6\ fm$.



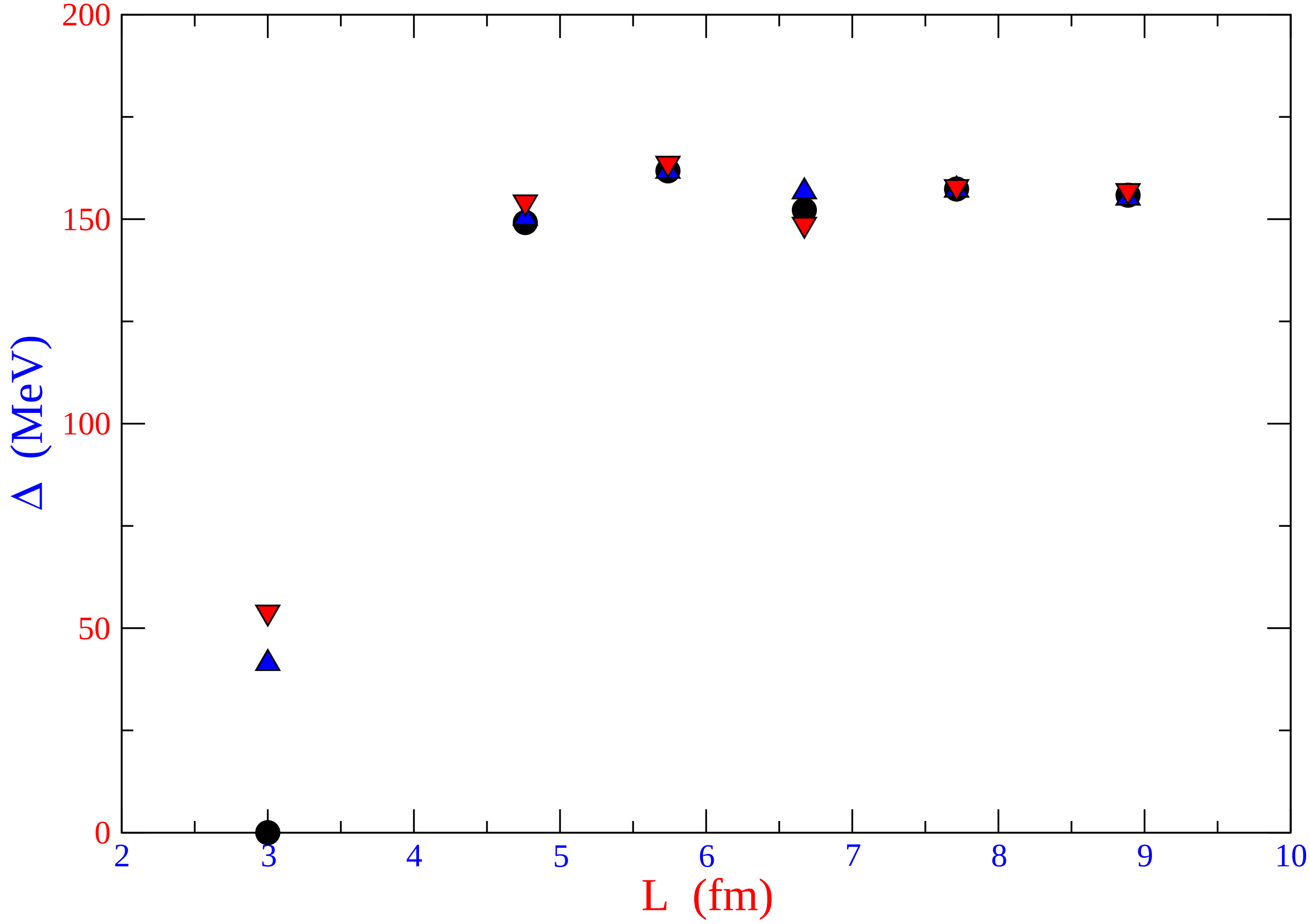
$$|\Psi\rangle_{BCS} = \sum_n \sqrt{d_n}\, |\Psi\rangle_n \qquad \sum_n d_n = 1$$



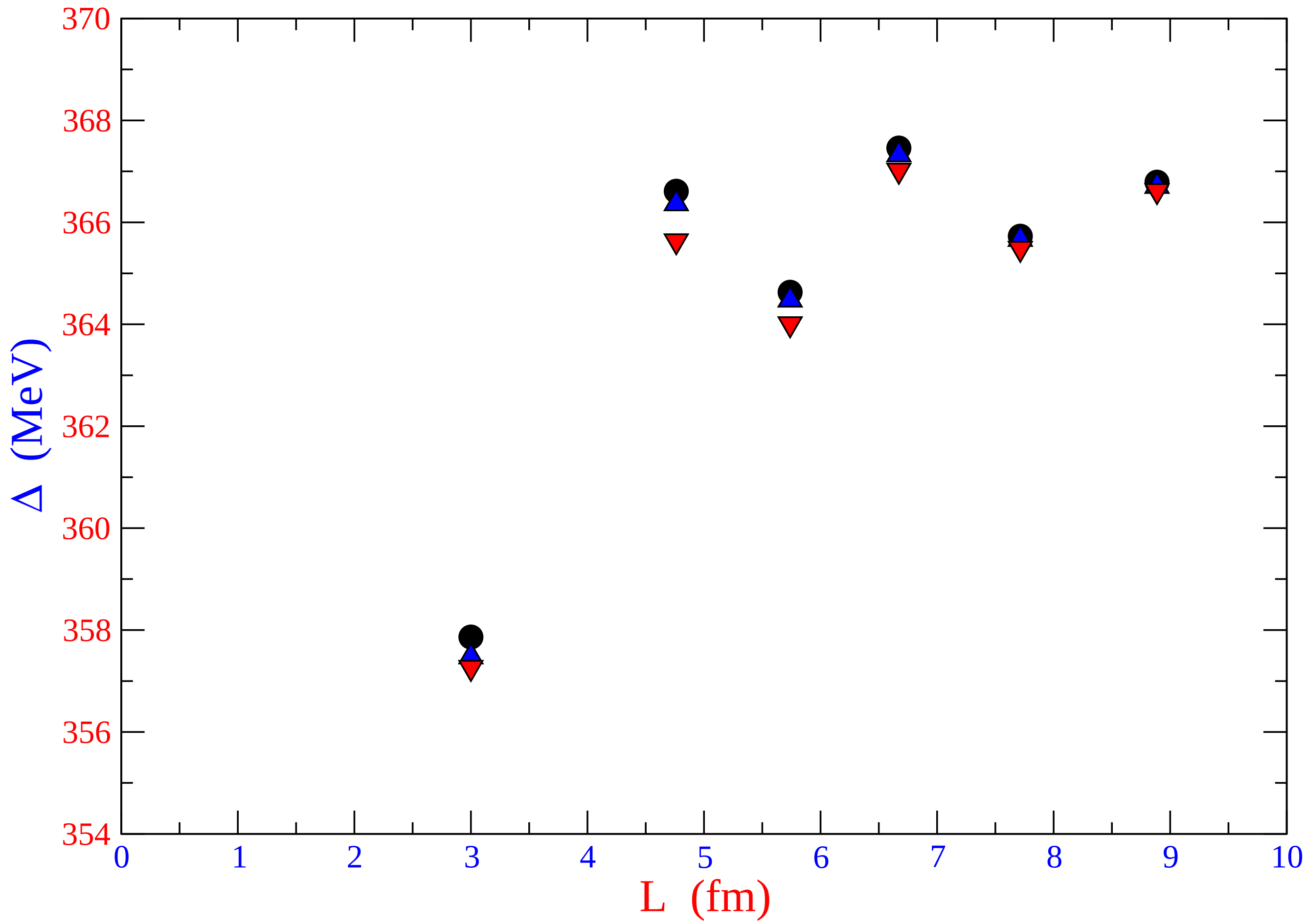
2SC gap for $L = 3\ fm$



2SC gap for $L = 6\ fm$



Behaviour of the 2SC gap at a fixed density changing the dimensions of the box



Behaviour of the energy at a fixed density changing the dimensions of the box

CONCLUSIONS

- We have used a phenomenological model with quarks, in the presence of 2 flavours, to study finite size effect on the color superconducting state
- The $2SC$ is projected **exactly over color and baryon number**
- The finite size effects turn out to be important for small systems (e.g. $L \approx 3\ fm$)
- We get large contributions from baryon number and color projection in this regime
- We are currently extending this calculation to study the $2SC$ condensate in presence of spontaneous chiral symmetry breaking

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